RETICULATE: Session Types as Algebraic Reticulates

Lattice Verification for Session Types with Parallel Composition

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Outline

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The Problem

Session types for objects

A session type describes the valid protocol for interacting with an object: which methods to call, in what order, under what conditions.

Limitation: existing systems assume *linear ownership* — exactly one client at a time.

Reality: concurrent access is everywhere:

- File I/O: concurrent readers and writers
- AI agents: tool calls $\|$ notifications (MCP)
- CI/CD: parallel lint, test, build jobs
- WebSockets: send $\|$ receive

The Insight

Parallel composition ⇒ lattice structure

Adding a *parallel constructor* $S_1 \parallel S_2$ to session types forces the state space to form a **lattice**.

Without $\parallel$:
State space is a tree/DAG.
Lattice structure is trivial.

With $\parallel$:
State space becomes a *product lattice*.
Meets = forks, joins = synchronizations.

$$L(S_1 \parallel S_2) \;=\; L(S_1) \times L(S_2)$$

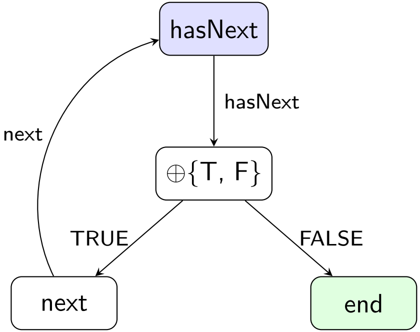
Grammar

$$
\begin{array}{rcll}
S & ::= & \&\{m_1 : S_1,\ \ldots,\ m_n : S_n\} & \text{branch (external choice)} \\
 & \mid & \oplus\{l_1 : S_1,\ \ldots,\ l_n : S_n\} & \text{selection (internal choice)} \\
 & \mid & S_1 \parallel S_2 & \textbf{parallel composition} \\
 & \mid & \mu X \,.\, S & \text{recursion} \\
 & \mid & X & \text{type variable} \\
 & \mid & \textbf{end} & \text{terminated} \\
 & \mid & S_1 \,.\, S_2 & \text{sequencing (sugar)}
\end{array}
$$

Key novelty: the $\parallel$ constructor — two concurrent sub-protocols on the same object, free interleaving until both reach **end**.

Example: Java Iterator

```
rec X . &{hasNext: ⊕{TRUE: &{next: X}, FALSE: end}}
```



- 4 states, 4 transitions
- 2 SCCs (cycle from recursion)
- Lattice: ✓
- Recursion visible as back-edge

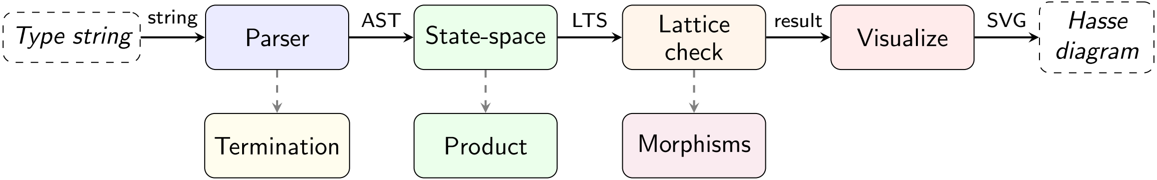
Example: Two-Buyer (with ∥)

```
lookup . getPrice . (proposeA . end ∥proposeB .
  ⊕{ACCEPT: pay . end, REJECT: end})
```

- Two buyers propose concurrently (∥).
- Buyer B may accept and pay, or reject.
- Product lattice: $2 \times 4 = 8$ states in the parallel region.
- **10 states, 14 transitions, lattice:** ✓

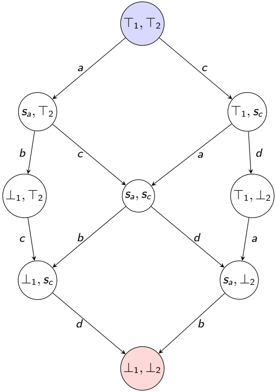
The ∥ constructor models the real concurrency: `proposeA` and `proposeB` may happen in any order.

Architecture



- 8 modules, **499 tests**, 17 benchmarks
- Python 3.12, pure library + CLI + web demo
- Product construction for $\parallel$ verified empirically on all benchmarks

Product Lattice



$a\,.\,b\,.\,\textbf{end} \;\|\; c\,.\,d\,.\,\textbf{end}$
$\Rightarrow\; 3 \times 3 = 9$ state product lattice.

- Top $= (\top_1, \top_2)$: fork point
- Bottom $= (\bot_1, \bot_2)$: join point
- Componentwise meets and joins

Lattice Verification

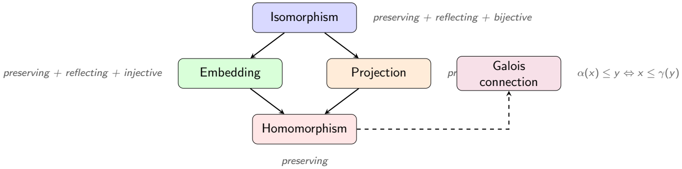
Given a state space $L(S)$, check the reachability poset:

1. **Quotient by SCCs** (Tarjan's) — handles cycles from μ
2. **Compute reachability** — topological-order DP
3. **Check four conditions:**

Condition	Meaning
Top reachable	$\top$ reaches all states
Bottom reachable	All states reach $\bot$
All meets exist	$\forall\, a, b :\ a \wedge b$ exists
All joins exist	$\forall\, a, b :\ a \vee b$ exists

Theorem: If $L(S_1)$ and $L(S_2)$ are lattices, then $L(S_1 \parallel S_2)$ is a lattice.

Morphism Hierarchy


Isomorphism
preserving + reflecting + bijective
preserving + reflecting + injective
Embedding
Projection
pr
Galois connection
$\alpha(x) \leq y \Leftrightarrow x \leq \gamma(y)$
Homomorphism
preserving

- Structure-preserving maps between state spaces
- **Isomorphism**: protocol equivalence
- **Embedding**: one protocol refines another
- **Galois connection**: abstraction–concretization (Cousot & Cousot)

17 Benchmark Protocols

#	Protocol	$\|Q\|$	$\|\delta\|$	SCCs	$\|$
1	Java Iterator	4	4	2	
2	File Object	5	5	4	
3	SMTP	7	8	4	
4	HTTP Connection	5	7	3	
5	OAuth 2.0	7	11	6	
6	Two-Buyer	10	14	10	✓
7	MCP (AI agent)	7	17	5	✓
8	A2A (Google agent)	5	7	4	
9	File Channel	7	12	7	✓
10	ATM	7	11	6	
11	Reentrant Lock	6	7	2	
12	WebSocket	6	15	6	✓
13	DB Transaction	5	9	3	
14	Pub/Sub	4	7	3	
15	DNS Resolver	4	8	2	
blue!8 16	**Reticulate Pipeline**	**14**	**18**	**14**	✓
blue!8 17	**GitHub CI Workflow**	**11**	**13**	**11**	✓

All 17 benchmarks produce lattices. 6 use $\|$.

Self-Reference: The Reticulate Pipeline

The reticulate web demo's own analysis pipeline as a session type:

```
input . parse .
  ⊕{OK: buildStateSpace .
     ⊕{OK: (checkLattice . checkTermination
             . checkWFPar . end
             ‖ renderDiagram . end)
            . returnResult . end,
        ERROR: end},
     ERROR: end}
```

- 14 states, 18 transitions
- Lattice: ✓
- WF-Par: ✓
- Terminating: ✓

The ‖ models real concurrency:

- Left: verification checks
- Right: diagram rendering
- Independent on shared state space

GitHub CI Workflow

A CI/CD pipeline — parallel jobs, branching on success/failure:

```
trigger . checkout . setup .
  (lint . end ∥ test . end) .
  ⊕{PASS: deploy . ⊕{OK: end,
                    FAIL: rollback . end},
    FAIL: end}
```

- 11 states, 13 transitions
- Lattice: ✓
- WF-Par: ✓
- Terminating: ✓

Natural ∥ usage:

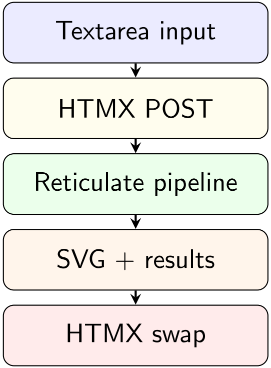
- `lint` and `test` run in parallel
- Fork-join before deployment
- Rollback on deploy failure

Session types capture the pipeline contract that developers already follow.

Interactive Web Tool

Stack: FastAPI + HTMX

- Enter any session type string
- Select from 17 benchmark protocols
- Instant results:
  - Pretty-printed type
  - State count, transitions, SCCs
  - Lattice verdict (4 sub-checks)
  - Termination & WF-Par status
  - Interactive SVG Hasse diagram
- Server-side rendering, zero JS overhead



Textarea input

HTMX POST

Reticulate pipeline

SVG + results

HTMX swap

Summary

Contributions

1. The $\|$ constructor for session types on objects — forces lattice structure
2. Reticulate: 8 modules, 499 tests, 17 benchmarks
3. Morphism hierarchy (iso $\subset$ embedding $\subset$ projection $\subset$ homomorphism) $+$ Galois connections
4. Web demo for instant, interactive analysis
5. Self-referential validation: the tool analyzes its own pipeline

Key result

All 17 benchmark protocols produce lattices — from classic patterns (Iterator, SMTP) through modern AI protocols (MCP, A2A) to the tool's own pipeline.

Future Work

- **Bica Reborn:** Java annotation-based session type checker, using RETICULATE as the verification back-end
- **Thread safety:** concurrency-level checking for methods in $\|$ branches (requires bytecode analysis)
- **Subtyping:** extending the lattice framework to session type subtyping with $\|$
- **Bisimulation:** connecting morphisms to behavioral equivalences on session types
- **Nested parallelism:** $\|$ inside $\|$ branches

Thank you

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Try it: `uvicorn web.app:app`

Reticulate — session types as algebraic reticulates